1. Given $y = x(2x + 1)^4$, show that

$$\frac{dy}{dx} = (2x + 1)^n(Ax + B)$$

where n, A and B are constants to be found.

(4)

If $y = uv$, $\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

$u = x$ $\quad v = (2x+1)^4$

$\frac{du}{dx} = 1$ $\quad \frac{dv}{dx} = 4(2x+1)^3 \times 2$ - (1)

$= 8(2x+1)^3$

$x \times 8(2x+1)^3 + (2x+1)^4 \times 1$

$= 8x(2x+1)^3 + (2x+1)^4$ - (1)

$= (2x+1)^3(8x + 2x + 1)$ - (1)

$= (2x+1)^3(10x+1)$

$n = 3$

$A = 10$ - (1)

$B = 1$

2.

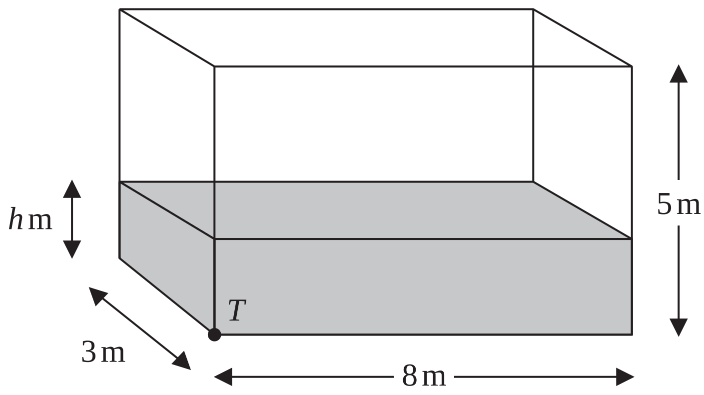

Figure 5

Water flows at a constant rate into a large tank.

The tank is a cuboid, with all sides of negligible thickness.

The base of the tank measures 8 m by 3 m and the height of the tank is 5 m.

There is a tap at a point T at the bottom of the tank, as shown in Figure 5.

At time t minutes after the tap has been opened

- the depth of water in the tank is h metres
- water is flowing into the tank at a constant rate of $0.48\,\text{m}^3$ per minute
- water is modelled as leaving the tank through the tap at a rate of $0.1h\,\text{m}^3$ per minute

(a) Show that, according to the model,

$$1200\frac{\mathrm{d}h}{\mathrm{d}t} = 24 - 5h$$

(4)

Given that when the tap was opened, the depth of water in the tank was 2 m,

(b) show that, according to the model,

$$h = A + B\mathrm{e}^{-kt}$$

where A, B and k are constants to be found.

(6)

Given that the tap remains open,

(c) determine, according to the model, whether the tank will ever become full, giving a reason for your answer.

(2)

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Question continued

a) $\frac{dV}{dt} = 0.48 - 0.1h$ ← change in volume with respect to time (1)

$V = 3 \times 8 \times h = 24h \quad \Rightarrow \quad \frac{dV}{dh} = 24$ (1)

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = (0.48 - 0.1h) \times \frac{1}{24}$

$= \frac{0.48 - 0.1h}{24}$ (1)

$24\frac{dh}{dt} = 0.48 - 0.1h$

× 50

$1200\frac{dh}{dt} = 24 - 5h$ □ (1)

b) $1200\frac{dh}{dt} = 24 - 5h \quad \Rightarrow \quad \frac{1200}{24 - 5h}\frac{dh}{dt} = 0$

$\int \left(\frac{1200}{24 - 5h} \times \frac{dh}{dt} \right) dt = \int dt$ (1) integrate both sides with respect to dt

$\int \frac{1200}{24 - 5h} \, dh = t \pm c$

integrate using substitution, or algebraic fractions 'shortcut' method

$-240\ln(24 - 5h) = t \pm c$ (1)

when $t = 0$, $h = 2$. substitute to find c.

$-240\ln(24 - 5(2)) \pm c = 0$

$-240\ln(14) \pm c = 0$

$\therefore c = +240\ln(14)$ (1)

Question continued

$$t = 240\ln(14) - 240\ln(24-5h) \quad ①$$

$$t = 240\ln\left(\frac{14}{24-5h}\right)$$

$$\frac{t}{240} = \ln\left(\frac{14}{24-5h}\right)$$

use inverse function to remove ln

$$e^{\frac{t}{240}} = \frac{14}{24-5h}$$

$$14e^{-\frac{t}{240}} = 24-5h \quad ①$$

$$h = 4.8 - 2.8e^{-\frac{t}{240}} \quad ①$$

c) Flow in = flow out at max. h, so
$0.1h = 4.8 \rightarrow h = 4.8.$ ① for finding max. h.

According to the model, the max. h is 4.8m but the tank is 5m tall, so the tank never becomes full. ①

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3. The curve C has parametric equations

$$x = 2\cos t, \quad y = \sqrt{3}\cos 2t, \quad 0 \leqslant t \leqslant \pi$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t.

(2)

The point P lies on C where $t = \frac{2\pi}{3}$

The line l is the normal to C at P.

(b) Show that an equation for l is

$$2x - 2\sqrt{3}y - 1 = 0$$

(5)

The line l intersects the curve C again at the point Q.

(c) Find the exact coordinates of Q.

You must show clearly how you obtained your answers.

(6)

a) $x = 2\cos t$ and $y = \sqrt{3}\cos(2t)$ $\quad \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ 1

$\frac{dx}{dt} = -2\sin t \qquad \frac{dy}{dt} = 2 \times \sqrt{3} \times -\sin(2t)$

$\frac{dy}{dt} = -2\sqrt{3}\sin(2t) \qquad \sin(2t) = 2\sin t\cos t$

$= \frac{dy}{dx} = \frac{-2\sqrt{3}\sin(2t)}{-2\sin(t)} = \frac{\sqrt{3}(2\sin t\cos t)}{\sin(t)} = \frac{2\sqrt{3}\cancel{\sin t}\cos t}{\cancel{\sin} t}$

$\Rightarrow \frac{dy}{dx} = \underline{2\sqrt{3}\cos(t)}$ 1

b) $\frac{dy}{dx} = 2\sqrt{3}\cos(t) = 2\sqrt{3}\cos\left(\frac{2\pi}{3}\right) = -\sqrt{3}$

Gradient of the normal $= -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \frac{1}{\sqrt{3}} = m$

$t = \frac{2\pi}{3}$ and $x = 2\cos t$ and $y = \sqrt{3}\cos(2t)$

$\Rightarrow x = 2\cos\left(\frac{2\pi}{3}\right)$ $\quad y = \sqrt{3}\cos\left(2\times\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

$\Rightarrow x = -1$

$y - \left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{\sqrt{3}}(x-(-1)) \Rightarrow y = \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}} - \frac{\sqrt{3}}{2}$

$\Rightarrow y = \frac{1}{\sqrt{3}}x - \frac{\sqrt{3}}{6}$

$\times\sqrt{3}$

$\Rightarrow \sqrt{3}y = x - \frac{1}{2}$

$\times 2$

$\Rightarrow 2\sqrt{3}y = 2x - 1$

$\Rightarrow 2x - 2\sqrt{3}y - 1 = 0$ as required.

c) $x = 2\cos t$ $\quad y = \sqrt{3}\cos(2t)$

Eq of line l : $2x - 2\sqrt{3}y - 1 = 0$

$\Rightarrow 2(2\cos t) - 2\sqrt{3}(\sqrt{3}\cos(2t)) - 1 = 0$ $\quad 6\cos(2t) = 6(2\cos^2 t - 1)$

$\Rightarrow 4\cos t - 6\cos(2t) - 1 = 0$ $\quad = 12\cos^2 t - 6$

$\Rightarrow 4\cos t - 12\cos^2 t + 6 - 1 = 0$

$\times -1$

$\Rightarrow 12\cos^2 t - 4\cos t - 5 = 0$

Now, let $\theta = \cos t$

$12\theta^2 - 4\theta - 5 = 0 \Rightarrow \theta = \frac{4 \pm \sqrt{(-4)^2 - 4(12)(-5)}}{2\times 24}$

+ve $\sqrt{\ }$: $\theta = \frac{5}{6}$, -ve $\sqrt{\ }$: $\theta = -\frac{1}{2}$

$\cos t = \frac{5}{6}$ $\quad \cos t = -\frac{1}{2} \Rightarrow t = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \Rightarrow$ ignore solution.

$x = 2\cos t$ and $y = \sqrt{3}\cos(2t)$ $\quad 2t = \cos^{-1}(5/6)\times 2$

$x = 2\times\frac{5}{6} = \frac{5}{3}$ $\quad y = \sqrt{3}\cos(\cos^{-1}(5/6)\times 2) = \frac{7\sqrt{3}}{18}$

$Q : \left(\frac{5}{3}, \frac{7}{18}\sqrt{3}\right)$